MATH 141 Midterm 2

November 15, 2022

NAME (please print legibly): Solution

Your University ID Number: ____________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ____________________

Enter your answers where indicated in order to receive credit. Calculators and notes are not permitted. If you are confused about the wording of a question or need a clarification, you should raise your hand and **ask a proctor** about it.

Unless otherwise indicated, you must show all work to justify your answers and receive full credit.

1. (15 points)

Let

$$f(x) = \frac{2x^k + x^3 - 2}{3x^4 - 2x}.$$

(a) Find all values of k so that $\lim_{x\to\infty} f(x) = 0$, or explain why this is not possible.

$$f(x) = \frac{x^k\left(2 + \frac{x^3}{x^k} - \frac{2}{x^k}\right)}{x^4\left(3 - \frac{2x}{x^4}\right)}$$

$$= x^{k-4}\left(\frac{2 + \frac{x^3}{x^k} - \frac{2}{x^k}}{3 - \frac{2}{x^3}}\right)$$

Then the limit at ∞ of f depends on whether $k-4>0$, $k-4<0$, or $k-4=0$.

If $k-4<0$, the limit will be zero.

Answer:

$k<4$

(b) Find all values of k so that $\lim_{x\to\infty} f(x) = \infty$, or explain why this is not possible.

Using the work in part (a), we can see that if $k-4>0$, we'll have an infinite limit.

Answer:

$k>4$

(c) Find all values of k so that $\lim_{x\to\infty} f(x) = \frac{2}{3}$, or explain why this is not possible.

Again using (a), if $k-4=0$,

Then

$$\lim_{x\to\infty} x^{k-4}\left(\frac{2 + \frac{x^3}{x^4} - \frac{2}{x^4}}{3 - \frac{2}{x^3}}\right)$$

$$= \lim_{x\to 0} \frac{2 + \frac{1}{x} - \frac{2}{x^4}}{3 - \frac{2}{x^3}} = \frac{2}{3}$$

Answer:

$k=4$

2. (10 points) Suppose we use the following limit to determine the derivative of a function $f(x)$ at $x = a$.

$$\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}.$$

(a) What is the function $f(x)$ and the number a?

The definition of the derivative is as follows:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a).$$

Equating like parts, $f(a+h) = \sqrt{4+h}$ and $f(a) = 2$.

Hence $f(x) = \sqrt{x}$, and $a = 4$.

Answer:
$f(x) = \sqrt{x}$
$a = 4$

(b) Find the derivative $f'(a)$ using any method you wish.

We could take this limit, but it's easier to use the power rule.

$$f(x) = x^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$
$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

Answer:
$f'(4) = \frac{1}{4}$

3. (20 points) Suppose that the functions f and g satisfy the following:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	$-\pi$	-2	2	3
2	1	7	5	4

(a) Let $h(x) = 3f(x) - 3g(x) + 3$. Find $h'(2)$.

$h'(x) = 3f'(x) - 3g'(x) + 0$

$h'(2) = 3(5) - 3(4) = 3$

Answer: 3

(b) Let $h(x) = \dfrac{f(x)}{g(x)}$. Find $h'(2)$.

We'll need the quotient rule.

$h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$h'(2) = \dfrac{(7)(5) - (1)(4)}{49} = \dfrac{81}{49}$

Answer: $\dfrac{31}{49}$

(c) Let $h(x) = (g \circ f)(x)$. Find $h'(2)$.

Using the chain rule

$h'(x) = g'(f(x))f'(x)$

$h'(2) = g'(f(2))f'(2)$

$= g'(1)(5)$

$= (3)(5)$

Answer: 15

(d) Let $h(x) = e^{f(x)g(x)}$. Find $h'(2)$.

We'll need the chain rule and the product rule.

$h'(x) = e^{f(x)g(x)}[f'(x)g(x) + g'(x)f(x)]$

$h'(2) = e^{(1)(7)}[(5)(7) + (4)(1)]$

$= e^7(39)$

Answer: $39e^7$

4. (10 points) Determine the derivative of the following functions. You do not have to simplify. Circle your final answer.

(a) $f(x) = e^x \ln(x) + \tan(x)$

$f'(x) = e^x \left(\frac{1}{x}\right) + (\ln x)(e^x) + \sec^2(x)$

(b) $h(z) = 2^z + z^{\sqrt{2}} + e^2 + 2^\pi$

$h'(z) = 2^z \ln 2 + \sqrt{2}\, z^{\sqrt{2}-1} + 0$

5. (20 points) Determine the derivatives of the following functions. You do not have to simplify. Circle your final answer.

(a) $f(x) = 5e^{x^3} + \ln(\ln x)$

$f'(x) = 5e^{x^3}(3x^2) + \frac{\frac{1}{x}}{\ln x}$

$= 15e^{x^3}x^2 + \frac{1}{x\ln x}$

(b) $g(t) = \ln(te^{-2t})$

$= \ln t + \ln(e^{-2t})$

$= \ln t - 2t$

$g'(t) = \frac{1}{t} - 2$

(c) $h(z) = (5z^2 - 6z)^9(z^3 + 7)$

$h'(z) = (5z^2-6z)^9(3z^2) + 9(5z^2-6z)^8(10z-6)(z^3+7)$

(d) $k(w) = \cos(\sqrt{w^5 + 3w^2})$

$k'(w) = -\sin(\sqrt{w^5+3w^2})\left(\frac{1}{2}\right)(w^5+3w^2)^{-\frac{1}{2}}(5w^4+6w)$

6. (15 points) Determine an equation for the line tangent to the curve satisfying

$$x^2 + xy = x + 3\sin(y)$$

at the point $(1, 0)$.

Taking the derivative implicitly,

$$2x + xy' + y = 1 + 3\cos(y)\,y'$$

Let $x=1$, $y=0$

$$2 + y' + 0 = 1 + 3(1)y'$$

$$1 = 2y'$$

$$y' = \frac{1}{2}$$

Line: $y - 0 = \frac{1}{2}(x-1)$

or $y = \frac{1}{2}x - \frac{1}{2}$

Answer:

$$y = \frac{1}{2}x - \frac{1}{2}$$

7. (10 points) Let $g(x) = (x^2+1)^x$. Find $g'(x)$. (Hint: Use logarithmic differentiation.)

$$\ln(g(x)) = \ln\left[(x^2+1)^x\right]$$
$$= x\ln(x^2+1)$$

Taking the derivative implicitly:

$$\frac{g'(x)}{g(x)} = x\left(\frac{2x}{x^2+1}\right) + \ln(x^2+1)$$

$$g'(x) = g(x)\left[\frac{2x^2}{x^2+1} + \ln(x^2+1)\right]$$
$$= (x^2+1)^x\left(\frac{2x^2}{x^2+1} + \ln(x^2+1)\right)$$

Answer:

$$g'(x) = (x^2+1)^x\left(\frac{2x^2}{x^2+1} + \ln(x^2+1)\right)$$